Statistica

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ES 14.1 (Libro Biagini)

Gli eventi $E_1, E_2, \dots$ sono stocasticamente indipendenti subordinatamente alla conoscenza del parametro Θ con $P(E_i \mid \Theta = \theta) = \theta$.

La densità a priori di Θ è data da

$$\pi_0 = \begin{cases} 3\theta^2 & 0 \le \theta \le 1 \\ 0 & \text{altrimenti} \end{cases}$$

Si osservano i valori dei primi 4 eventi:

$$E_1 = 0$$
$$E_2 = 1$$
$$E_3 = 1$$
$$E_4 = 1$$

1) Calcolare la densità a posteriori di Θ

2) Calcolare la probabilità a priori che Θ appartenga all'intervallo $[\frac{1}{2}, 1]$

3) Calcolare la probabilità a posteriori che Θ appartenga all'intervallo $[\frac{1}{2}, 1]$

4) Calcolare il valore massimo della densità a posteriori di Θ

5) Calcolare la previsione a posteriori di $E = E_5 \wedge E_6$

SOL

1) La densità a posteriori si calcola usando la formula

$$\pi_4(\Theta=\theta \mid E_1=0, E_2=1, E_3=1, E_4=1) =$$

$$= K \cdot \pi_0(\theta) \cdot P(E_1=0, E_2=1, E_3=1, E_4=1 \mid \Theta=\theta) =$$

Posso scomporre $P(E_1=0, E_2=1, E_3=1, E_4=1 \mid \Theta=\theta)$ perché E_1, E_2, E_3, E_4 sono indipendenti

$$= K \cdot \pi_0(\theta) \cdot P(E_1=0 \mid \Theta=\theta) \cdot P(E_2=1 \mid \Theta=\theta) \cdot P(E_3=1 \mid \Theta=\theta) \cdot P(E_4=1 \mid \Theta=\theta) =$$

$$= K \cdot \pi_0 \cdot (1-\theta) \cdot \theta \cdot \theta \cdot \theta \overset{\pi_0 = 3\theta^2}{=}$$

$$= K \cdot \pi_0 \cdot (1-\theta) \cdot \theta^3 = K \cdot 3\theta^2 \cdot (1-\theta) \cdot \theta^3 =$$

$$= 3K \cdot \theta^5 (1-\theta)$$

Resta da calcolare K: Dobbiamo sfruttare la relazione

$$1 = \int_{\mathbb{R}} \pi_4 \, d\theta$$

$$\Rightarrow \quad 1 = \int_0^1 \pi_4 \, d\theta$$

$$\Rightarrow \quad 1 = \int_0^1 3K\theta^5(1-\theta) \, d\theta$$

$$\Rightarrow \quad 1 = 3K \int_0^1 \theta^5 - \theta^6 \, d\theta$$

$$\Rightarrow \quad 1 = 3K \left(\frac{\theta^6}{6} - \frac{\theta^7}{7}\right)\Bigg|_0^1 = 3K\left(\frac{1}{6} - \frac{1}{7}\right) = 3K \frac{1}{42} = \frac{K}{14}$$

$$\Rightarrow \quad K = 14$$

$$\Rightarrow \quad \pi_4 = 3 \cdot 14 \cdot \theta^5(1-\theta) = 42\,\theta^5(1-\theta)$$

2) La probabilità a priori che θ appartenga a $[\frac{1}{2},1]$ è:

$$P\left(\frac{1}{2}\leq\theta\leq1\right)=\int_{\frac{1}{2}}^{1}\pi_0\,d\theta=\int_{\frac{1}{2}}^{1}3\theta^2\,d\theta=\theta^3\Big|_{\frac{1}{2}}^{1}=$$

$$=1-\frac{1}{8}=\frac{7}{8}$$

3) La probabilità a posteriori che θ appartenga a $[\frac{1}{2},1]$ è:

$$P\left(\frac{1}{2}\leq\theta\leq1\,\middle|\,E_1=0,E_2=1,E_3=1,E_4=1\right)=\int_{\frac{1}{2}}^{1}\pi_4\,d\theta=$$

$$=\int_{\frac{1}{2}}^{1}42\,\theta^5(1-\theta)\,d\theta=42\int_{\frac{1}{2}}^{1}\theta^5-\theta^6\,d\theta=$$

$$=42\left(\frac{\theta^6}{6}-\frac{\theta^7}{7}\right)\Big|_{\frac{1}{2}}^{1}=42\left(\frac{1}{6}-\frac{1}{7}-\left(\frac{1}{6}\cdot\frac{1}{2^6}-\frac{1}{7}\cdot\frac{1}{2^7}\right)\right)=\frac{15}{16}$$

4) Il valore massimo della densità a posteriori si trova derivando $\pi_4(\theta)$ e ponendo uguale a zero tale derivata:

$$\frac{\partial}{\partial\theta}\pi_4(\theta)=42\cdot\left(5\theta^4(1-\theta)+\theta^5(-1)\right)=$$

$$=42\left(5\theta^4(1-\theta)-\theta^5\right)$$

$\Rightarrow$

$$42\left(5\theta^4(1-\theta)-\theta^5\right)=0\iff5\theta^4(1-\theta)-\theta^5=0$$

$$\iff5\theta^4-5\theta^5-\theta^5=0\iff5\theta^4=6\theta^5\iff\theta=\frac{5}{6}$$

$$\Leftrightarrow 5\theta^4 - 5\theta^5 - \theta^5 = 0 \Leftrightarrow 5\theta^4 = 6\theta^5 \Leftrightarrow \theta = \frac{5}{6}$$

$$\Rightarrow \bar{\theta} = \frac{5}{6}$$

5) La previsione a posteriori di $E = E_5 \wedge E_6 = \min(E_5, E_6) =$
$= E_5 E_6$

coincide con la sua probabilità a posteriori

Dunque

$$P(E_5 E_6 \mid E_1 = 0, E_2 = 1, E_3 = 1, E_4 = 1) = \int_0^1 P(E_5 E_6 \mid \theta) \cdot P(\theta \mid E_1 = 0, E_2 = 1, E_3 = 1, E_4 = 1)\, d\theta =$$

$$= \int_0^1 P(E_5 E_6 \mid \theta) \cdot \pi_4 \, d\theta = \int_0^1 P(E_5 \mid \theta) \cdot P(E_6 \mid \theta) \cdot \pi_4 \, d\theta =$$

$$= \int_0^1 P(E_5 \mid \theta)\, P(E_6 \mid \theta) \cdot 42\,\theta^5 (1-\theta)\, d\theta =$$

$$= \int_0^1 \theta \cdot \theta \cdot 42 \cdot \theta^5 (1-\theta)\, d\theta = 42 \int_0^1 \theta^7 (1-\theta)\, d\theta =$$

$$= 42 \int_0^1 \theta^7 - \theta^8 \, d\theta = 42 \left(\frac{\theta^8}{8} - \frac{\theta^9}{9} \right) \Bigg|_0^1 = 42 \left(\frac{1}{8} - \frac{1}{9} \right) =$$

$$= 42 \cdot \frac{1}{72} = \frac{7}{12}$$

<u>ES 14.2</u>

Gli eventi $E_1, E_2, \dots$ sono stocasticamente indipendenti subordinatamente alla conoscenza del parametro

Gli eventi $E_1, E_2, \ldots$ sono stocasticamente indipendenti subordinatamente alla conoscenza del parametro aleatorio Θ con $P(E_i | \Theta = \theta) = \theta$

La densità a priori di Θ è data da

$$\pi_0(\theta) = \begin{cases} K\,\theta^2(1-\theta) & 0 \le \theta \le 1 \\ 0 & \text{altrimenti} \end{cases}$$

Si osservano i valori dei primi cinque eventi

$E_1 = 0$
$E_2 = 1$
$E_3 = 1$
$E_4 = 0$
$E_5 = 1$

1) Calcolare K
2) Calcolare la densità a posteriori di Θ e la probabilità a posteriori dell'evento $(\Theta < \frac{1}{2})$
3) Calcolare la previsione a posteriori di $X = E_6 + E_7$ e le probabilità a posteriori di $E = E_6 E_7$ e $F = E_6 \vee E_7$

SOL

1) Per calcolare K si impone che l'integrale della densità sia pari ad 1, cioè:

$$1 = \int_{\mathbb{R}} \pi_0(\theta)\, d\theta$$

$\Rightarrow$

$$1 = \int_0^1 \pi_0(\theta)\, d\theta$$

$$\Rightarrow 1 = \int_0^1 k\,\theta^2(1-\theta)\,d\theta$$

$$\Rightarrow 1 = k\left(\frac{\theta^3}{3} - \frac{\theta^4}{4}\right)\Big|_0^1 = k\left(\frac{1}{3}-\frac{1}{4}\right) = k\left(\frac{1}{12}\right)$$

$$\Rightarrow k = 12$$

$$\Rightarrow \quad \pi_0(\theta) = \begin{cases} 12\,\theta^2(1-\theta) & 0 \le \theta \le 1 \\ 0 & \text{altrove} \end{cases}$$

2) La densità a posteriori è data da:

$$\pi_5(\Theta=\theta \mid E_1=0, E_2=1, E_3=1, E_4=0, E_5=1) =$$

$$= K\, \pi_0(\theta)\cdot P(E_1=0, E_2=1, E_3=1, E_4=0, E_5=1 \mid \Theta=\theta) =$$

$$= K\cdot\pi_0\cdot P(E_1=0\mid\Theta=\theta)\cdot P(E_2=1\mid\Theta=\theta)\cdot P(E_3=1\mid\Theta=\theta)\cdot P(E_4=0\mid\Theta=\theta)\cdot P(E_5=1\mid\Theta=\theta) =$$

$$= K\cdot\pi_0\cdot(1-\theta)\cdot\theta\cdot\theta\cdot(1-\theta)\cdot\theta =$$

$$= K\cdot\pi_0\cdot\theta^3(1-\theta)^2 = K\cdot 12\,\theta^2(1-\theta)\cdot\theta^3(1-\theta)^2 =$$

$$= K\cdot 12\,\theta^5(1-\theta)^3$$

Troviamo K:

$$1 = \int_{\mathbb{R}} \pi_4(\theta)\,d\theta = \int_0^1 12K\,\theta^5(1-\theta)^3\,d\theta =$$

$$= 12K\,\frac{\Gamma(6)\cdot\Gamma(4)}{\Gamma(10)} = 12K\,\frac{5!\,3!}{9!} =$$

$$= 12K\,\frac{1}{9\cdot 8\cdot 7} = K\cdot\frac{1}{42}$$

$$\Rightarrow K = 42$$

$$\Rightarrow \pi_5 = 42 \cdot 12 \cdot \theta^5 (1-\theta)^3 = 504\, \theta^5 (1-\theta)^3$$

Per trovare la probabilità a posteriori si procede così:

$$P\left(\theta < \tfrac{1}{2}\right) = \int_0^{1/2} \pi_5(\theta)\, d\theta =$$

$$= \int_0^{1/2} 504\, \theta^5 (1-\theta)^3\, d\theta =$$

$$(1-\theta)^3 = (1 - \theta^3 - 3\theta + 3\theta^2) \cdot \theta^5$$
$$= \theta^5 - \theta^8 - 3\theta^6 + 3\theta^7$$

$$= 504 \int_0^{1/2} \theta^5 (1-\theta)^2 (1-\theta)\, d\theta =$$

$$= 504 \int_0^{1/2} \theta^5 (1-\theta)\,(1+\theta^2-2\theta)\, d\theta =$$

$$= 504 \int_0^{1/2} (\theta^5 - \theta^6)(1+\theta^2-2\theta)\, d\theta =$$

$$= 504 \int_0^{1/2} \theta^5 + \theta^7 - 2\theta^6 - \theta^6 - \theta^8 + 2\theta^7\, d\theta =$$

$$= 504 \left(\frac{\theta^6}{6} + \frac{\theta^8}{8} - 2\frac{\theta^7}{7} - \frac{\theta^7}{7} - \frac{\theta^9}{9} + 2\frac{\theta^8}{8}\right)\Bigg|_0^{1/2} =$$

$$= 504 \left(\frac{1}{6}\cdot\frac{1}{2^6} + \frac{1}{8}\cdot\frac{1}{2^8} - \frac{2}{7}\cdot\frac{1}{2^7} - \frac{1}{7}\cdot\frac{1}{2^7} - \frac{1}{9}\cdot\frac{1}{2^9} + \frac{2}{8}\cdot\frac{1}{2^8}\right) =$$

$$= 504 \left(\frac{1}{384} + \frac{1}{2048} - \frac{1}{448} - \frac{1}{896} - \frac{1}{4608} + \frac{1}{2048}\right) =$$

$$= \frac{55}{256}$$

3) Sia $X = E_6 + E_7$

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La previsione a posteriori di X è data da:

sono 2 eventi aggiuntivi

$$P(X|\Psi) = \sum_{i=0}^{2} i \, P(X=i|\Psi) =$$

$$= P(X=1|\Psi) + 2\,P(X=2|\Psi)$$

Dobbiamo calcolare dunque $P(X=1|\Psi)$ e $P(X=2|\Psi)$:

- $P(X=1|\Psi) = P(E_6+E_7=1|\Psi) =$

$$= P(E_6=1, E_7=0|\Psi) + P(E_6=0, E_7=1|\Psi) =$$

$$= \int_0^1 \pi_5(\theta) \cdot \left(P(E_6=1, E_7=0|\Psi) + P(E_6=0, E_7=1|\Psi)\right) d\theta$$

$$= \int_0^1 504 \cdot \theta^5 (1-\theta)^3 \left(P(E_6=1|\Psi) \cdot P(E_7=0|\Psi) + P(E_6=0|\Psi) \cdot P(E_7=1|\Psi)\right) d\theta =$$

$$= \int_0^1 504\, \theta^5 (1-\theta)^3 \left(\theta(1-\theta) + (1-\theta) \cdot \theta\right) d\theta =$$

$$= 504 \int_0^1 \theta^5 (1-\theta)^3 \cdot \left(2\theta(1-\theta)\right) d\theta =$$

$$= 504 \cdot 2 \int_0^1 \theta^6 (1-\theta)^4 \, d\theta =$$

$$= 1008 \int_0^1 \theta^6 (1-\theta)^4 \, d\theta =$$

$$= 1008 \cdot \frac{\Gamma(7)\,\Gamma(5)}{\Gamma(12)} = 1008 \cdot \frac{6! \cdot 4!}{11!} =$$

$$= \frac{24}{55}$$

$$= \frac{24}{55}$$

$$\cdot\ P(X=2 \mid \psi) = P(E_6 + E_7 = 2 \mid \psi) = P(E_6 = 1, E_7 = 1 \mid \psi)$$

$$= \int_0^1 \pi_5 \cdot P(E_6 = 1, E_7 = 1 \mid \psi)\, d\theta =$$

$$= \int_0^1 504\, \theta^5 (1-\theta)^3 \left(P(E_6 = 1 \mid \psi) \cdot P(E_7 = 1 \mid \psi)\right) d\theta =$$

$$= 504 \int_0^1 \theta^5 (1-\theta)^3 \cdot \theta^2\, d\theta =$$

$$= 504\, \frac{\Gamma(8) \cdot \Gamma(4)}{\Gamma(12)} = 504\, \frac{7! \cdot 3!}{11!} = \frac{21}{55}$$

In conclusione

$$P(X \mid \psi) = \frac{24}{55} + 2 \cdot \frac{21}{55} = \frac{66}{55} = \frac{6}{5}$$

Calcoliamo le probabilità a posteriori di $E = E_6 E_7$ e $F = E_6 \vee E_7$:

$$\cdot\ P(E_6 E_7 = 1 \mid \psi) = P(E_6 = 1, E_7 = 1 \mid \psi) =$$

SALTO I SOLITI PASSAGGI FORMALI

$$= \int_0^1 \pi_5\, \theta^2\, d\theta = 504 \int_0^1 \theta^5 (1-\theta)^3\, \theta^2\, d\theta =$$

$$= 504\, \frac{\Gamma(8)\, \Gamma(4)}{\Gamma(12)} = \frac{21}{55}$$

$$\bullet\ P(F|\psi) = P(E_6 \vee E_7 = 1 \,|\, \psi) =$$

$$= P(E_6=1, E_7=0|\psi) + P(E_6=0, E_7=1|\psi) + P(E_6=1, E_7=1|\psi) =$$

SALTO I PASSAGGI

$$= \int_0^1 \pi_5 \left(\theta(1-\theta) + (1-\theta)\cdot\theta + \theta^2\right) d\theta =$$

$$= 504 \int_0^1 \theta^5 (1-\theta)^3 \left(2\theta(1-\theta) + \theta^2\right) d\theta =$$

$$= 504 \int_0^1 2\theta^6 (1-\theta)^4 + \theta^7 (1-\theta)^3 \, d\theta =$$

$$= 504 \cdot 2 \frac{\Gamma(7)\Gamma(5)}{\Gamma(12)} + 504 \frac{\Gamma(8)\Gamma(4)}{\Gamma(12)} =$$

$$= 1008 \cdot \frac{6!\,4!}{11!} + 504 \frac{7!\,3!}{11!} = \frac{9}{11}$$

$\blacksquare$

ES 14.3

Gli eventi $E_1, E_2, \dots$ sono stocasticamente indipendenti subordinatamente alla conoscenza del parametro aleatorio Θ con $P(E_i|\Theta=\theta)=\theta$

La densità a priori di Θ è data da

$$\pi_0(\theta) = \begin{cases} K\,\theta^2(1-\theta)^2 & 0 \le \theta \le 1 \\ 0 & \text{altrove} \end{cases}$$

Si osservano i valori dei primi 4 eventi

$E_1 = 0$
$E_2 = 1$
$E_3 = 1$
$E_4 = 1$

a) Calcolare la costante K
b) Calcolare la densità e la previsione a posteriori di Θ
c) Calcolare la probabilità a posteriori dell'evento $F = E_5^2$
e la varianza a posteriori dell'evento $\tilde{E}_6$

SOL

a) Sfruttiamo la relazione

$$1 = \int_0^1 \pi_0(\theta)\, d\theta = \int_0^1 K\, \theta^2 (1-\theta)^2\, d\theta =$$

$$= K\, \frac{\Gamma(3)\,\Gamma(3)}{\Gamma(6)} = K\, \frac{2!\ 2!}{5!} = K\, \frac{4}{5!} = K\, \frac{1}{5 \cdot 3 \cdot 2}$$

$\Rightarrow$

$K = 30$

$$\Rightarrow \pi_0(\theta) = 30\, \theta^2 (1-\theta)^2 \quad \text{se} \quad 0 \leq \theta \leq 1$$

b) La densità a posteriori è data da:

$$\pi_4(\Theta = \theta \mid E_1 = 0, E_2 = 1, E_3 = 1, E_4 = 1) =$$

$$= K\, \pi_0(\theta) \cdot P(E_1 = 0, E_2 = 1, E_3 = 1, E_4 = 1 \mid \Theta = \theta) =$$

$$= K\, \pi_0(\theta) \cdot (1-\theta) \cdot \theta^3 = K \cdot 30\, \theta^2 (1-\theta)^2 \cdot (1-\theta)\, \theta^3 =$$

$$= 30 K\, \theta^5 (1-\theta)^3$$

$$= 30K\,\theta^5(1-\theta)^3$$

Troviamo K:

$$1 = \int_0^1 \pi_4(\theta)\, d\theta = \int_0^1 30K\,\theta^5(1-\theta)^3\, d\theta =$$

$$= 30K\,\frac{\Gamma(6)\,\Gamma(4)}{\Gamma(10)} = 30K\,\frac{5!\;3!}{9!} = K\,\frac{5}{84}$$

$$\Rightarrow K = \frac{84}{5}$$

$$\Rightarrow \pi_4 = 504 \cdot \theta^5(1-\theta)^3$$

Previsione a posteriori:
Per calcolare la previsione a posteriori di Θ si applica la formula della previsione per le variabili aleatorie con distribuzione assolutamente continua, cioè:

$$P(\Theta \mid E_1=0, E_2=1, E_3=1, E_4=1) = \int_0^1 \theta\,\pi_4(\theta)\, d\theta =$$

$$= \int_0^1 \theta\, 504\,\theta^5(1-\theta)^3\, d\theta =$$

$$= 504 \int_0^1 \theta^6(1-\theta)^3\, d\theta =$$

$$= 504\,\frac{\Gamma(7)\,\Gamma(4)}{\Gamma(11)} = 504\,\frac{6!\cdot 3!}{10!} = \frac{3}{5}$$

c) L'evento $F = E_5^2$ coincide con E_5

Dunque

Dunque

$$P(F \mid E_1=0, E_2=1, E_3=1, E_4=1) = P(E_5^2 \mid E_1=0, E_2=E_3=E_4=1) =$$

$$= P(E_5 \mid E_1=0, E_2=1, E_3=1, E_4=1) =$$

$$= \int_0^1 P(E_5 \mid \theta) \cdot \pi_4(\theta)\, d\theta = \int_0^1 \theta \cdot \pi_4(\theta)\, d\theta = \dots = \frac{3}{5}$$

Varianza a posteriori:

Per calcolare la varianza a posteriori dell'evento $\tilde{E}_6$, si considera per prima cosa la solita formula della varianza; si ha:

$$Var(\tilde{E}_6 \mid E_1=0, E_2=1, E_3=1, E_4=1) =$$

$$= P(\tilde{E}_6^2 \mid E_1=0, E_2=1, E_3=1, E_4=1) - \left(P(\tilde{E}_6 \mid E_1=0, E_2=1, E_3=1, E_4=1)\right)^2 =$$

$$= P(\tilde{E}_6 \mid E_1=0, E_2=1, E_3=1, E_4=1) - \left(P(\tilde{E}_6 \mid E_1=0, E_2=1, E_3=1, E_4=1)\right)^2 =$$

metto in evidenza $P(\tilde{E}_6 \mid \dots)$

$$= P(\tilde{E}_6 \mid E_1=0, E_2=E_3=E_4=1)\left(1 - P(\tilde{E}_6 \mid \dots)\right)$$

Dobbiamo calcolare $P(\tilde{E}_6 \mid \dots)$ (La previsione a posteriori)

Per calcolare $P(\tilde{E}_6 \mid \dots)$ osserviamo che non conosciamo $\tilde{E}_5$

Dobbiamo usare allora la formula delle probabilità totali (a posteriori):

$$P(\tilde{E}_6 \mid \dots) = P(\tilde{E}_6 E_5 \mid \dots) + P(\tilde{E}_6 \tilde{E}_5 \mid \dots) =$$

$$= \int^1 P(\tilde{E}_6 E_5 \mid \dots) \cdot \pi_4(\theta)\, d\theta + \int^1 P(\tilde{E}_6 \tilde{E}_5 \mid \dots) \cdot \pi_4(\theta)\, d\theta =$$

$$= \int_0^1 P(\tilde{E}_6 E_5 \mid \dots) \cdot \pi_4(\theta)\, d\theta + \int_0^1 P(\tilde{E}_6 \tilde{E}_5 \mid \dots) \cdot \pi_4(\theta)\, d\theta =$$

$$= \int_0^1 P(\tilde{E}_6 \mid \dots) \cdot P(E_5 \mid \dots) \cdot \pi_4(\theta)\, d\theta + \int_0^1 P(\tilde{E}_6 \mid \dots) \cdot P(\tilde{E}_5 \mid \dots) \cdot \pi_4(\theta)\, d\theta =$$

$$= \int_0^1 (1-\theta) \cdot \theta \cdot \pi_4(\theta)\, d\theta + \int_0^1 (1-\theta)(1-\theta) \cdot \pi_4(\theta)\, d\theta =$$

$$= \int_0^1 504\, \theta^6 (1-\theta)^4\, d\theta + \int_0^1 504\, \theta^5 (1-\theta)^5\, d\theta =$$

$$= 504 \frac{\Gamma(7)\Gamma(5)}{\Gamma(12)} + 504 \frac{\Gamma(6)\Gamma(6)}{\Gamma(12)} = \frac{12}{55} + \frac{2}{11} = \frac{2}{5}$$

$\Rightarrow$

$$Var(\tilde{E}_6 \mid \dots) = \frac{2}{5}\left(1 - \frac{2}{5}\right) = \frac{6}{25}$$

ES 14.4

Gli eventi $E_1, E_2, \dots$ sono stocasticamente indipendenti subordinatamente alla conoscenza di un parametro aleatorio Θ con $P(E_1 \mid \Theta = \theta) = \theta$

La densità a priori di Θ è data da

$$\pi_0(\theta) = \begin{cases} K\, \theta^2 \sqrt{1-\theta} & 0 \le \theta \le 1 \\ 0 & \text{altrimenti} \end{cases}$$

Si osservano i primi 4 eventi:

$E_1 = 1$

Si osservano i seguenti 4 eventi:

$$E_1 = 1$$
$$E_2 = 0$$
$$E_3 = 0$$
$$E_4 = 1$$

a) Calcolare K

b) Calcolare la densità a posteriori di Θ ed il suo punto di massimo

c) Calcolare la covarianza a posteriori degli eventi E_6 e E_7

<u>SOL</u>

a) Come al solito

$$1 = \int_0^1 \pi_0(\theta)\, d\theta = \int_0^1 K\theta^2\sqrt{1-\theta}\; d\theta = K\int_0^1 \theta^2(\sqrt{1-\theta})\, d\theta$$

$$= K\cdot\frac{4}{35} \Rightarrow K = \frac{35}{4}$$

$$\int \underbrace{x^2}_{f(x)}\underbrace{\sqrt{1-x}}_{g'(x)\,\Rightarrow\, g(x) = -\frac{2}{3}(1-x)^{3/2}}\, dx = x^2\left(-\frac{2}{3}(1-x)^{3/2}\right) - \int 2x\left(-\frac{2}{3}(1-x)^{3/2}\right) dx =$$

$$= -\frac{2}{3}x^2(1-x)^{3/2} + \frac{4}{3}\int x(1-x)^{3/2}\, dx =$$

$$= -\frac{2}{3}x^2(1-x)^{3/2} + \frac{4}{3}\left(x\left(-\frac{2}{5}\right)(1-x)^{5/2} - \int -\frac{2}{5}(1-x)^{5/2}\, dx\right) =$$

$$= -\frac{2}{3}x^2(1-x)^{3/2} - \frac{8}{15}x(1-x)^{5/2} + \frac{2}{5}\int (1-x)^{5/2}\, dx =$$

$$= -\frac{2}{3}x^2(1-x)^{3/2} - \frac{8}{15}x(1-x)^{5/2} + \frac{2}{5}\cdot\left(-\frac{2}{7}(1-x)^{7/2}\right) =$$

$$= -\frac{2}{3}x^2(1-x)^{3/2} - \frac{8}{15}x(1-x)^{5/2} - \frac{4}{35}(1-x)^{7/2}$$

$$\Rightarrow \pi_0 = \frac{35}{4}\theta^2\sqrt{1-\theta}$$

b) Come al solito:

$$\pi_4(\Theta=\theta \mid E_1=1, E_2=0, E_3=0, E_4=1) = K\, \pi_0(\theta) \cdot P(E_1=1, E_2=0, E_3=0, E_4=1 \mid \Theta=\theta) =$$

$$= K\, \pi_0(\theta)\ \theta^2(1-\theta)^2 = K \cdot \frac{35}{4}\theta^2\sqrt{1-\theta} \cdot \theta^2(1-\theta)^2 =$$

$$= \frac{35}{4}K\ \theta^4(1-\theta)^{5/2}$$

Troviamo K:

$$1 = \int_0^1 \pi_4(\theta)\, d\theta = \int_0^1 \frac{35}{4}K\theta^4(1-\theta)^{5/2}\, d\theta = \frac{35}{4}K\int_0^1 \theta^4(1-\theta)^{5/2}\, d\theta =$$

$$= \frac{35}{4}K\ \frac{\Gamma(5)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(5+\frac{7}{2}\right)} = \frac{35}{4}K\ \frac{\Gamma(5)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{17}{2}\right)} =$$

Sfrutto $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

$$= \frac{35}{4}K\ \frac{\Gamma(5)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(1+\frac{15}{2}\right)} = \frac{35}{4}K\ \frac{\Gamma(5)\Gamma\left(\frac{7}{2}\right)}{\frac{15}{2}\cdot\Gamma\left(\frac{15}{2}\right)} =$$

$$= \frac{35}{4}K \cdot \frac{\Gamma(5)\Gamma\left(\frac{7}{2}\right)}{\frac{15}{2}\cdot\frac{13}{2}\Gamma\left(\frac{13}{2}\right)} = \frac{35}{4}K\ \frac{\Gamma(5)\cdot\Gamma\left(\frac{7}{2}\right)}{\frac{15}{2}\cdot\frac{13}{2}\cdot\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\,\Gamma\left(\frac{7}{2}\right)} =$$

$$= \frac{35}{4}K \cdot \frac{\Gamma(5)}{\frac{15}{2}\cdot\frac{13}{2}\cdot\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}} = \frac{64}{1287}K$$

$$\Rightarrow K = \frac{1287}{64}$$

$$\Rightarrow \pi_4 = \frac{1287}{64}\cdot\frac{35}{4}\ \theta^4(1-\theta)^{5/2}$$

Il punto di max si trova calcolando gli zeri della sua derivata prima:

$$\frac{\partial}{\partial\theta}\pi_4 = 4\theta^3(1-\theta)^{5/2} + \theta^4\left(\frac{5}{2}(1-\theta)^{3/2}(-1)\right) =$$

$$\frac{\partial}{\partial\theta}\pi_4 = 4\theta^{\cdot}(1-\theta) + \theta\,(2 \ldots$$

$$= 4\theta^3(1-\theta)^{5/2} - \frac{5}{2}\theta^4(1-\theta)^{3/2}$$

$\Rightarrow$

$$4\theta^3(1-\theta)^{5/2} - \frac{5}{2}\theta^4(1-\theta)^{3/2} = 0 \iff$$

$$\iff 1 = \frac{5}{2}\cdot\frac{1}{4}\,\theta\,(1-\theta)^{-1} \iff \frac{8}{5} = \frac{\theta}{1-\theta} \iff$$

$$\iff \frac{8}{5} - \frac{8}{5}\theta = \theta \iff \theta + \frac{8}{5}\theta = \frac{8}{5} \iff \frac{13}{5}\theta = \frac{8}{5}$$

$$\iff \theta = \frac{8}{13}$$

$$\Rightarrow \bar{\theta} = \frac{8}{13}$$

c) La covarianza a posteriori degli eventi E_6 e E_7
è data da:

$$\mathrm{Cov}(E_6, E_7 \mid E_1 = 1, E_2 = 0, E_3 = 0, E_4 = 1) =$$

$$= P(E_6 E_7 \mid \ldots) - P(E_6 \mid \ldots)\cdot P(E_7 \mid \ldots)$$

Dobbiamo perciò calcolare $P(E_6 \mid \ldots)$, $P(E_7 \mid \ldots)$ e $P(E_6 E_7 \mid \ldots)$

- $P(E_6 \mid \ldots) = P(E_6 E_5 \mid \ldots) + P(E_6 \tilde{E}_5 \mid \ldots) =$

$$= \int_0^1 P(E_6 E_5 \mid \ldots)\cdot \pi_4(\theta)\,d\theta + \int_0^1 P(E_6 \tilde{E}_5 \mid \ldots)\cdot \pi_4(\theta)\,d\theta =$$

$$= \int_0^1 \theta^2 \cdot \pi_4\,d\theta + \int_0^1 \theta(1-\theta)\,\pi_4(\theta)\,d\theta =$$

$$= \int_0^1 \pi_4(\theta)\,(\theta^2 + \theta - \theta^2)\,d\theta = \int_0^1 \theta\,\pi_4(\theta)\,d\theta =$$

il momento "quello lì"

il numero "quello lì"

$$= \int_0^1 \theta \cdot c\, \theta^4 (1-\theta)^{5/2}\, d\theta = c \int_0^1 \theta^5 (1-\theta)^{5/2}\, d\theta =$$

$$= c\, \frac{\Gamma(6)\, \Gamma(\frac{7}{2})}{\Gamma(\frac{19}{2})} = c\, \frac{\Gamma(6) \cdot \Gamma(\frac{7}{2})}{\frac{17}{2}\,\frac{15}{2}\,\frac{13}{2}\,\frac{11}{2}\,\frac{9}{2}\,\frac{7}{2}\,\Gamma(\frac{7}{2})} = \frac{10}{17}$$

- $P(E_7|\ldots) = P(E_7 E_6|\ldots) + P(E_7 \tilde{E}_6|\ldots) =$

$$= \int_0^1 P(E_7|\ldots)\, P(E_6|\ldots)\, \pi_4(\theta)\, d\theta + \int_0^1 P(E_7|\ldots) \cdot P(\tilde{E}_6|\ldots) \cdot \pi_4 =$$

$$= \int_0^1 \theta\, \pi_4(\theta)\, d\theta = \ldots = \frac{10}{17}$$

- $P(E_6 E_7|\ldots) = P(E_6 E_7 E_5|\ldots) + P(E_6 E_7 \tilde{E}_5|\ldots) =$

$$= \int_0^1 P(E_6 E_7 E_5|\ldots)\, \pi_4 + \int_0^1 P(E_6 E_7 \tilde{E}_5|\ldots) \cdot \pi_4\, d\theta =$$

$$= \int_0^1 \theta^3 \pi_4 + \int_0^1 \theta^2 (1-\theta)\, \pi_4\, d\theta =$$

$$= \int_0^1 \pi_4 \left(\theta^3 + \theta^2 - \theta^3\right) d\theta =$$

$$= \int_0^1 \pi_4 \cdot \theta^2\, d\theta = \int_0^1 \theta^2 \cdot c \cdot \theta^4 (1-\theta)^{5/2}\, d\theta =$$

$$= c \int_0^1 \theta^6 (1-\theta)^{5/2}\, d\theta = c\, \frac{\Gamma(7)\, \Gamma(\frac{7}{2})}{\Gamma(\frac{21}{2})} =$$

$$= c\, \frac{\Gamma(7)\, \Gamma(\frac{7}{2})}{\frac{19}{2}\,\frac{17}{2}\,\frac{15}{2}\,\frac{13}{2}\,\frac{11}{2}\,\frac{9}{2}\,\frac{7}{2}\,\Gamma(\frac{7}{2})} = \frac{120}{323}$$

$$= c \; \frac{\Gamma(7)\,\Gamma(\frac{7}{2})}{\frac{19}{2}\cdot\frac{17}{2}\cdot\frac{15}{2}\,\frac{13}{2}\,\frac{11}{2}\,\frac{9}{2}\cdot\frac{7}{2}\,\Gamma(\frac{7}{2})} = \frac{720}{323}$$

In conclusione

$$Cov\left(E_6, E_7 \mid E_1 = 1, E_2 = E_3 = 0, E_4 = 1\right) = \frac{720}{323}\cdot\left(\frac{10}{17}\right)^2$$

ES 14.5

Le variabili aleatorie $X_1, X_2, \ldots$ sono stocasticamente indipendenti subordinatamente alla conoscenza del parametro Θ con densità subordinata marginale

$$f(x_i \mid \theta) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(x_i-\theta)^2}{2}} \qquad x_i \in R$$

La distribuzione a priori di Θ è una normale standard.

Si osservano i valori dei primi 4 esperimenti:

$x_1 = 0{,}1$

$x_2 = 2$

$x_3 = -1$

$x_4 = 0{,}5$

a) Scrivere la densità a priori di Θ

b) Calcolare la densità a posteriori di Θ ed il suo punto di massimo

c) Calcolare previsione e varianza a posteriori di Θ

SOL

a) Poiché Θ ha distribuzione a priori di normale standard, si può scrivere subito la sua densità a priori:

$$\pi_0(\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}} \qquad \theta \in \mathbb{R}$$

b) Per calcolare la densità a posteriori di Θ si usa il fatto che le variabili aleatorie sono stocasticamente indipendenti.

Dunque

$$\pi_4(\theta \mid x_1 = 0{,}1,\ x_2 = 2,\ x_3 = -1,\ x_4 = 0{,}5) = K \cdot \pi_0(\theta) \cdot f(x_1, x_2, x_3, x_4 \mid \theta) =$$

$$= K \cdot \pi_0 \cdot \prod_{i=1}^{4} f(x_i \mid \theta) = K \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}} \cdot \prod_{i=1}^{4} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-\theta)^2}{2}} =$$

$$= K \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\sum_{i=1}^{4}(x_i-\theta)^2}{2}} =$$

$$= K \cdot \frac{1}{2\pi} e^{-\frac{\sum_{i=1}^{4}(x_i-\theta)^2 + \theta^2}{2}} = K \cdot \frac{1}{2\pi} e^{-\frac{(0{,}1-\theta)^2 + (2-\theta)^2 + (-1-\theta)^2 + (0{,}5-\theta)^2 + \theta^2}{2}} =$$

$$= K \cdot \frac{1}{2\pi} e^{-\frac{\frac{1}{100} + \theta^2 - 0{,}2\theta + 4 + \theta^2 - 4\theta + 1 + \theta^2 + 2\theta + \frac{1}{4} + \theta^2 - \theta + \theta^2}{2}} =$$

$$= K \cdot \frac{1}{2\pi} e^{-\frac{\frac{263}{50} + 5\theta^2 - \frac{14}{5}\theta}{2}} =$$

$$= K \frac{1}{2\pi} e^{-\frac{263}{100}} \cdot e^{-\frac{5}{2}\theta^2 + \frac{14}{10}\theta} = \tilde{K} \cdot e^{\left(-\frac{5}{2}\theta^2 + \frac{14}{10}\theta\right)}$$

2π

Troviamo il max :

$$\frac{d}{d\theta}\pi_4 = e^{-\frac{5}{2}\theta^2+\frac{14}{10}\theta}\left(-5\theta+\frac{14}{10}\right) = 0 \iff$$

$$\iff -5\theta = -\frac{14}{10} \iff \theta = \frac{14}{50} = \frac{7}{25}$$

$$\Rightarrow \bar{\theta} = \frac{7}{25}$$

c) La previsione:

$$P(\Theta \mid x_1=0{,}1,\ x_2=2,\ x_3=-1,\ x_4=0{,}5) = ???$$

La varianza:

$$\sigma^2(\Theta \mid x_1=0{,}1,\ x_2=2,\ x_3=1,\ x_4=0{,}5) = ???$$

ES 14.6

Le variabili aleatorie $X_1, X_2, \dots$ sono stocasticamente indipendenti subordinatamente alla conoscenza del parametro Θ con densità subordinata marginale

$$f(x_i|\theta) = \frac{1}{2\sqrt{2\pi}}\, e^{-\frac{(x_i-\theta)^2}{8}}$$

La densità a priori di Θ è data da

La densità a priori di Θ è data da

$$\pi_0(\theta) = \frac{1}{\sqrt{4\pi}} e^{-\frac{(\theta-1)^2}{4}}$$

Si osservano i valori dei primi tre esperimenti:

$$x_1 = 1$$
$$x_2 = 0,5$$
$$x_3 = -1$$

a) Calcolare il fattore di verosimiglianza

b) Calcolare la densità a posteriori di Θ

c) Stimare la probabilità a posteriori dell'evento $(\Theta > 1000)$

<u>SOL</u>

a) Il fattore di verosimiglianza è per definizione dato da:

$$f(x_1,x_2,x_3|\theta) = \prod_{i=1}^{3} f(x_i|\theta) =$$

$$= \prod_{i=1}^{3} \frac{1}{2\sqrt{2}} e^{-\frac{(x_i-\theta)^2}{8}} = \left(\frac{1}{2\sqrt{2\pi}}\right)^3 \prod_{i=1}^{3} e^{-\frac{(x_i-\theta)^2}{8}} =$$

$$= \left(\frac{1}{2\sqrt{2\pi}}\right)^3 e^{-\frac{\sum_{i=1}^{3}(x_i-\theta)^2}{8}} = \frac{1}{2\sqrt{2}} e^{-\frac{(1-\theta)^2+(0,5-\theta)^2+(-1-\theta)^2}{8}} =$$

$$= \left(\frac{1}{2\sqrt{2\pi}}\right)^3 e^{-\frac{1+\theta^2-2\theta+\frac{1}{4}+\theta^2-\theta+1+\theta^2+2\theta}{8}} =$$

$$= \left(\frac{1}{2\sqrt{2\pi}}\right)^3 e^{-\frac{\frac{9}{4}+3\theta^2-\theta}{8}} = \frac{1}{8\sqrt{(2\pi)^3}} e^{-\frac{9}{32}+\frac{3}{8}\theta^2-\frac{1}{8}\theta}$$

$$= \left(\frac{1}{2\sqrt{2\pi}}\right) e^{-\frac{\quad}{8}} = \frac{1}{8\sqrt{(2\pi)^3}}\, e$$

$$= \frac{1}{8\sqrt{(2\pi)^3}}\, e^{-\frac{1}{8}\left(\frac{9}{4}+3\theta^2-\theta\right)}$$

b) Dal calcolo del fattore di verosimiglianza si ricava subito la densità a posteriori nel seguente modo:

$$\pi_3(\theta \mid x_1=1, x_2=0{,}5, x_3=-1) = K \cdot \pi_0(\theta) \cdot f(x_1,x_2,x_3 \mid \theta) =$$

$$= K \cdot \frac{1}{\sqrt{4\pi}}\, e^{-\frac{(\theta-1)^2}{4}} \cdot \frac{1}{8\sqrt{(2\pi)^3}}\, e^{-\frac{1}{8}\left(\frac{9}{4}+3\theta^2-\theta\right)} =$$

$$= \tilde{K}\, e^{-\frac{(\theta-1)^2}{4}-\frac{1}{8}\left(\frac{9}{4}+3\theta^2-\theta\right)} = \tilde{K}\, e^{-\frac{\theta^2+1-2\theta}{4}-\frac{9}{32}-\frac{3}{8}\theta^2+\frac{1}{8}\theta} =$$

$$= \tilde{K}\, e^{\frac{-8\theta^2-8+16\theta-9-12\theta^2+4\theta}{32}} =$$

$$= \tilde{K}\, e^{\frac{-20\theta^2+20\theta-17}{32}} = \tilde{K}\, e^{-\frac{5}{8}\theta^2+\frac{5}{8}\theta-\frac{17}{32}} =$$

$$= \tilde{K}\, e^{-\frac{5}{8}\left(\theta^2-\theta+\frac{17}{20}\right)}$$

c) ????